Topic 3 -
Linear first order
ODEs

Topic 3 - First order linear ODEs

We will give a method to solve

$$y' + a(x)y = b(x)$$

on any interval I where $a(x), b(x)$ are continuous.

Since $a(x)$ is continuous we can find an antiderivative

$$A(x) = \int a(x)\,dx$$

So, $A'(x) = a(x)$

Ex: $b(x) = x$

$$y' + \underbrace{2x}_{}y = \overbrace{x}^{}$$

$a(x) = 2x$

$I = (-\infty, \infty)$

Ex

$$A(x) = \int 2x\,dx$$
$$= x^2$$

Multiply $y' + a(x)y = b(x)$ by $e^{A(x)}$ to get:

$$\underbrace{e^{A(x)} y' + e^{A(x)} a(x) y}_{\left(e^{A(x)} \cdot y\right)'} = b(x) e^{A(x)}$$

Ex

$y' + 2xy = x$

$e^{x^2} y' + 2x e^{x^2} y = x e^{x^2}$

We get

$$\left(e^{A(x)} \cdot y\right)' = b(x) e^{A(x)}$$

Ex

$\left(e^{x^2} \cdot y\right)' = x e^{x^2}$

Integrate both sides with respect to x to get:

$$e^{A(x)} y = \int b(x) e^{A(x)} dx$$

Ex

$e^{x^2} y = \frac{1}{2} e^{x^2} + C$

Solve for y:

$$y = e^{-A(x)} \int b(x) e^{A(x)} dx$$

Since you can reverse the above steps this is the only solution to the ODE.

$$y = \frac{1}{2} \underbrace{e^{-x^2} e^{x^2}}_{e^{-x^2+x^2} = e^0 = 1} + Ce^{-x^2}$$

$$y = \frac{1}{2} + Ce^{-x^2}$$

Ex:

Solve
$$\frac{dy}{dx} + 2xy = xe^{-x^2}$$
on $I = (-\infty, \infty)$

We want to solve
$$y' + \underbrace{2x}_{\text{integrate this}}y = xe^{-x^2}$$

$$A(x) = \int 2x\,dx = x^2$$

Multiply the ODE by

$e^{A(x)} = e^{x^2}$ to get:

$$\underbrace{e^{x^2} y' + e^{x^2}(2x)y}_{\text{always } \left(e^{A(x)} \cdot y\right)'} = e^{x^2} \cdot x \cdot e^{-x^2}$$

This becomes:

$$\left(e^{x^2} \cdot y\right)' = \underbrace{e^{x^2} \cdot x \cdot e^{-x^2}}_{e^{x^2} e^{-x^2} = e^{x^2 - x^2} = e^0 = 1}$$

So we get

$$\left(e^{x^2} \cdot y\right)' = x$$

Integrate both sides with respect to x to get

$$e^{x^2} \cdot y = \int x\,dx$$

We get

$$e^{x^2} \cdot y = \frac{1}{2}x^2 + C$$

Divide by e^{x^2} or multiply by e^{-x^2} to get

$$\boxed{y = \frac{1}{2}x^2 e^{-x^2} + Ce^{-x^2}}$$

Answer

Ex: Let's solve

$$y' + \underbrace{\cos(x)}_{\text{integrate this}} y = \sin(x)\cos(x)$$

on $I = (-\infty, \infty)$

$$A(x) = \int \cos(x)\,dx = \sin(x)$$

Multiply the ODE by $e^{A(x)} = e^{\sin(x)}$ to get:

$$\underbrace{e^{\sin(x)} y' + e^{\sin(x)} \cos(x)\, y}_{\text{always } \left(e^{A(x)} \cdot y\right)'} = e^{\sin(x)} \sin(x)\cos(x)$$

We get:

$$\left(e^{\sin(x)} \cdot y\right)' = e^{\sin(x)} \sin(x)\cos(x)$$

Integrate both sides with respect to x to get

$$e^{\sin(x)} \cdot y = \int e^{\sin(x)} \sin(x)\cos(x)\,dx$$

$$\int e^{\sin(x)} \sin(x)\cos(x)\,dx$$

$$= \int e^{t} \cdot t\,dt = \int t e^{t}\,dt$$

$$= te^{t} - \int e^{t}\,dt$$

$t = \sin(x)$
$dt = \cos(x)dx$

LIATE

$u = t$ $\quad du = dt$
$dv = e^{t}dt$ $\quad v = e^{t}$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

$$= te^{t} - e^{t} + C$$

$$= \sin(x)e^{\sin(x)} - e^{\sin(x)} + C$$

Thus,

$$e^{\sin(x)} \cdot y = \sin(x)e^{\sin(x)} - e^{\sin(x)} + C$$

Divide by $e^{\sin(x)}$ or multiply by $e^{-\sin(x)}$ to get:

$$\cancel{e^{-\sin(x)}}\,\cancel{e^{\sin(x)}}\,y = \cancel{e^{-\sin(x)}}\sin(x)\cancel{e^{\sin(x)}} - \cancel{e^{-\sin(x)}}\,\cancel{e^{\sin(x)}} + Ce^{-\sin(x)}$$

We get:

$$\boxed{y = \sin(x) - 1 + Ce^{-\sin(x)}}$$

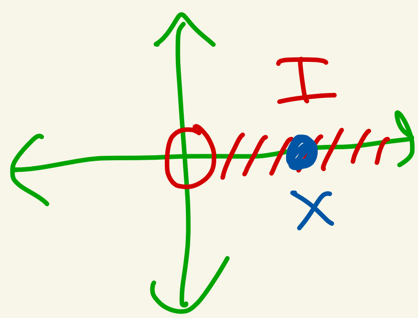
Answer

$$I = (-\infty, \infty)$$

<u>Ex:</u> Solve

$$xy' + y = 3x^3 + 1$$

on $I = (0, \infty)$



The technique doesn't work with the x in front of y'.

Divide the ODE by x to get

$$y' + \underbrace{\frac{1}{x}}_{\text{integrate this}} y = 3x^2 + \frac{1}{x}$$

Let

$$A(x) = \int \frac{1}{x}\,dx = \ln|x|$$

$$= \ln(x)$$

$$\boxed{\begin{array}{l} I = (0, \infty) \\ \text{So, } x > 0 \end{array}}$$

Multiply the ODE by

$$e^{A(x)} = e^{\ln(x)} = x$$

$$\boxed{\begin{array}{l} e^{\ln(z)} = z \\ \text{for any } z > 0 \end{array}}$$

We get

$$\underbrace{x y' + y}_{\text{always is}} = 3x^3 + 1$$

$$\left(e^{A(x)} \cdot y\right)'$$

This becomes:

$$(xy)' = 3x^3 + 1$$

Integrate with respect to x to get

$$xy = \int (3x^3 + 1)\, dx$$

We get

$$xy = \frac{3}{4}x^4 + x + C$$

Divide by x to get

$$y = \frac{3}{4}x^3 + 1 + \frac{C}{x}$$

So all sols to $\boxed{A}$

$$xy' + y = 3x^3 + 1$$

on $I = (0, \infty)$

are of the form

$$y = \frac{3}{4}x^3 + 1 + \frac{c}{x}$$

ANSWER

Ex: Solve

$$xy' + y = 3x^3 + 1$$ ODE

$$y(1) = 2$$ Condition on solution

on $I = (0, \infty)$

We already know that

$$y = \frac{3}{4}x^3 + 1 + \frac{c}{x}$$

is the general solution to

$$xy' + y = 3x^3 + 1.$$

Let's make $y(1) = 2$.

We need

$$\underbrace{\frac{3}{4}(1)^3 + 1 + \frac{C}{1}}_{y(1)} = 2$$

We get $\frac{7}{4} + C = 2$

So, $C = 2 - \frac{7}{4} = \frac{1}{4}$

$\boxed{\text{Answer: } y = \frac{3}{4}x^3 + 1 + \frac{1/4}{x}}$